

QUOTIENTS BY TORSION ELEMENTS

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ABSTRACT. We construct a two generator recursively presented group with infinite torsion length. We also explore the construction in the context of solvable and word-hyperbolic groups.

1. INTRODUCTION

What should the ‘torsion’ subgroup of an arbitrary group be? The set of torsion elements in a group does not work; it is not necessarily a subgroup. To be more precise, the product of two torsion elements need not be torsion; for an easy example of this, consider the group $C_2 * C_2$. Attempting to consider the subgroup generated by the set of torsion elements as the ‘torsion’ subgroup of a group is also unsatisfactory; the quotient of a group by this subgroup need not be torsion free, as shown in proposition 4.1. We can, however, iterate this procedure: letting $\text{Tor}_1(G)$ be the subgroup generated by the torsion elements of a group G , we inductively define $\text{Tor}_{n+1}(G)/\text{Tor}_n(G) = \text{Tor}_1(G/\text{Tor}_n(G))$ (where it can be seen that $\text{Tor}_n(G) \trianglelefteq G$), and form the union $\text{Tor}_\infty(G) = \bigcup_{n \in \mathbb{N}} \text{Tor}_n(G)$. The subgroup $\text{Tor}_\infty(G)$ is a viable candidate for the ‘torsion’ subgroup of a group: it coincides with the classical notion of torsion subgroup when the set of torsion elements do form a subgroup, $G/\text{Tor}_\infty(G)$ is torsion-free, and every homomorphism from G to a torsion-free group annihilates $\text{Tor}_\infty(G)$. All of this is described in greater detail in §3.

The structure of $\text{Tor}_\infty(G)$ as a countable union of subgroups allows us to attach an invariant to any group G , which we call the *torsion length* of G and denote by $\text{TorLen}(G)$. A group is said to have torsion length n (definition 3.9) if n is the smallest natural number such that $\text{Tor}_\infty(G) = \text{Tor}_n(G)$; the torsion length is said to be infinite if no such natural number exists.

We now give a quick overview of the contents of this paper. In §2, we describe some notation, and give some basic group-theoretic lemmas that we will need for this paper. As noted earlier, §3 concerns itself with describing elementary aspects of our iterated torsion subgroups and the notion of torsion length.

We mentioned earlier that there are groups such that $G/\text{Tor}_1(G)$ is not torsion-free. In §4, we extend the construction given in proposition 4.1 to

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construct finitely presented groups with arbitrary finite torsion length. More precisely, we prove the following result (writing \overline{P} to denote the group presented by a presentation P):

Theorem 4.3. *There is a family of finite presentations $\{P_n\}_{n \in \mathbb{N}}$ such that:*

1. $\overline{P}_n / \langle\langle \text{Tor}(\overline{P}_n) \rangle\rangle \cong \overline{P}_{n-1}$.
2. $\text{TorLen}(\overline{P}_n) = n$.

The examples constructed in the above result are not pathological: we show in proposition 4.5 that they are all word hyperbolic. The proof is an application of a result of Kharlampovich and Myasnikov [5].

The group-theoretic construction of the \overline{P}_n above was also presented independently by Cirio *et. al.* [3, Example 5.16], albeit without providing an explicit finite presentation. Moreover, Leary and Nucinkis [6, §5 Corollary 7] give an alternative group-theoretic construction, which we describe in theorem 4.7. We elected to include both constructions, as each has benefit over the other. For example, the presentations we give in theorem 4.3 have $2^n - 1$ generators and $2^n - 1$ relators, but all relators are length 3 or 5, and are easily indexed and described; the presentations of Leary and Nucinkis in theorem 4.7 have $2n - 1$ generators and at most $2n - 1$ relators, but the relators are extremely complicated).

In §5 and §6, we turn our attention to groups of infinite torsion length. There is a well-known, uniform process for embedding a countable group into a two generator group. We describe this process and some of its immediate consequences in some detail in lemma 5.1 and lemma 5.2. The remainder of §5 concerns itself with verifying that the procedure described in lemma 5.1 does not change torsion length:

Theorem 5.7. *Let P be a countably generated recursive presentation (respectively, finite presentation). Then we can construct a 2-generator recursive presentation (respectively, finite presentation) $\text{fg}(P)$, uniformly in P , such that \overline{P} embeds in $\text{fg}(\overline{P})$, and $\text{TorLen}(\text{fg}(\overline{P})) = \text{TorLen}(\overline{P})$.*

Using this, it is not too difficult to prove our main result:

Theorem 6.3. *There exists a 2-generator recursive presentation Q for which $\text{TorLen}(\overline{Q}) = \infty$.*

The following is naturally of interest to us; we do not know the answer to it.

Question. *Does there exist a finitely presented group of infinite torsion length?*

If G is nilpotent, it is well known [7, 5.2.7] that the product of two torsion elements is indeed torsion, and thus that the set of torsion elements forms a subgroup. In particular, every nilpotent group has torsion length at most 1. In §7, we show that this is not necessarily the case for polycyclic groups:

Corollary 7.3. *There is a finitely presented polycyclic group of torsion length 2.*

Unfortunately, we have been unable to construct finitely presented solvable groups of torsion length greater than two. We consider the existence of these objects an interesting question for future research:

Question. *Do there exist polycyclic, or at the very least finitely generated solvable, groups of torsion length n for arbitrary n ?*

One can also ask if there exist finitely generated solvable groups of infinite torsion length: such an example could not have the maximum condition on normal subgroups, and in particular could not be polycyclic. Again, we do not know whether such a group exists.

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2. PRELIMINARIES

2.1. Notation. If $P = \langle X|R \rangle$ is a group presentation with generating set X and relators R , then we denote by \overline{P} the group presented by P . A presentation $P = \langle X|R \rangle$ is said to be a *recursive presentation* if X is a finite set and R is a recursive enumeration of relators; P is said to be a *countably generated recursive presentation* if instead X is a recursive enumeration of generators. A group G is said to be *finitely* (respectively, *recursively*) *presentable* if $G \cong \overline{P}$ for some finite (respectively, recursive) presentation P . If P, Q are group presentations then we denote their free product presentation by $P * Q$; this is given by taking the disjoint union of their generators and relators. If g_1, \dots, g_n are a collection of elements of a group G , then we write $\langle g_1, \dots, g_n \rangle$ for the subgroup in G generated by these elements; however, we may write $\langle g_1, \dots, g_n \rangle^G$ when we want to clarify what the ambient group is. As is standard, we write $\langle\langle g_1, \dots, g_n \rangle\rangle^G$ for the normal closure of these elements in G . Let $\text{o}(g)$ denote the order of a group element g .

2.2. Preliminary facts in group theory. In this section, we collect a few lemmas that we will need later in this paper. They all must be well known, but we have been unable to find suitable references.

Lemma 2.1. *Let G_1 and G_2 be non-trivial groups, and suppose that G_1 has order strictly greater than 2. Then, $G_1 * G_2$ contains a non-abelian free subgroup.*

Proof. Since G_1 has order greater than 2, there exist y and z such that $yz \neq e$. Let x be a non trivial element of G_2 . The reader will easily check that the elements yxz and $xyxz$ freely generate a free subgroup in $G_1 * G_2$. \square

Lemma 2.2. *$C_2 * C_2$ is polycyclic*

Proof. Let the two copies of C_2 be generated by x and y respectively. Then, since $xyx = yx$ and $yxy = yx$, it follows that the cyclic subgroup generated

by xy is normal. It is also easy to see that $C_2 * C_2 / \langle xy \rangle \cong C_2$. Thus $C_2 * C_2$ is polycyclic. \square

Definition 2.3 ([5]). A subgroup H of a group G is said to be *conjugate separated* if for any $x \in G \setminus H$ we have that $H \cap xHx^{-1}$ is finite.

Lemma 2.4. *Let A and B be groups, and suppose $e \neq a \in A$ and $e \neq b \in B$, with either $\text{o}(a) \neq 2$ or $\text{o}(b) \neq 2$. Then for any $x \in A * B \setminus \langle ab \rangle$, $\langle ab \rangle \cap x\langle ab \rangle x^{-1} = \{e\}$. Hence $\langle ab \rangle$ is conjugate separated in $A * B$.*

Proof. Without loss of generality we may take $\text{o}(b) \neq 2$.

Suppose that $\langle ab \rangle$ is not conjugate separated in $A * B$. Then, there exists an $x \in A * B \setminus \langle ab \rangle$, and $i, j \in \mathbb{Z} \setminus \{0\}$ such that $x(ab)^i x^{-1} = (ab)^j$. We can assume that the underlying word of x is reduced. We will induct on the length of x as a reduced word.

Let us first assume that $i > 0$.

It follows that there must exist $x' \in A * B$ such that either $x = x'a^{-1}$, or $x = x'b$. This is true because any other eventuality would lead to $x(ab)^i x^{-1}$ having a reduced underlying word with begins and ends with a letter from the same group, and an element with such a word clearly cannot belong to $\langle ab \rangle$.

Since $\text{o}(b) \neq 2$, $\langle ab \rangle \cap \langle ba \rangle = \{e\}$. It follows that $x' \neq e$, as $a^{-1}\langle ab \rangle a = b\langle ab \rangle b^{-1} = \langle ba \rangle$. We therefore have that $x'(ba)^i x'^{-1} = (ab)^j$. Assume that $x = x'a^{-1}$, then by reasoning as we did the previous paragraph, we see that $x' = x''b^{-1}$ (the other option, where $x' = x''a$, cannot happen as that would then imply that x was not reduced.) It follows that $x = x''b^{-1}a^{-1}$. Assuming that $x = x'b$, a similar line of reasoning allows us to reach the conclusion that $x = x''ab$, for some element x'' . In either case, we have that $x''\langle ab \rangle^i x''^{-1} = (ab)^j$. Applying our induction hypothesis, we see that $x'' \in \langle ab \rangle$. Thus $x \in \langle ab \rangle$.

The case where $i < 0$ is analogous. \square

Remark 2.5. The converse to lemma 2.4 also holds: if both a and b have order 2, then the subgroup $\langle ab \rangle$ is not conjugate separated. To see this, observe that if both a and b have order 2, then $aaba = babb = ba = (ab)^{-1}$.

3. TORSION

We make the following two definitions; the latter first appeared in [2].

Definition 3.1. Let G be a group, and let $g \in G$. Recall that g is *torsion* if $1 \leq \text{o}(g) < \infty$. We then write the set of torsion elements of G as:

$$\text{Tor}(G) := \{g \in G \mid g \text{ is torsion}\}$$

Definition 3.2. Given a group G , we inductively define $\text{Tor}_i(G)$ as follows:

$$\text{Tor}_0(G) := \{e\}$$

$$\text{Tor}_{i+1}(G) := \langle \{g \in G \mid g \text{Tor}_i(G) \in \text{Tor}(G/\text{Tor}_i(G))\} \rangle^G$$

$$\text{Tor}_\infty(G) := \bigcup_{i \in \mathbb{N}} \text{Tor}_i(G)$$

Thus, $\text{Tor}_i(G)$ is the set of elements of G which are annihilated upon taking i successive quotients of G by the normal closure of all torsion elements, and $\text{Tor}_\infty(G)$ is the union of all these. By construction, $\text{Tor}_i(G) \leq \text{Tor}_j(G)$ whenever $i \leq j$. It follows immediately that $\text{Tor}_\infty(G) \trianglelefteq G$.

The following lemma also appears as [2, Proposition 4.9]. We give a proof here for the sake of completeness.

Lemma 3.3. *$G/\text{Tor}_\infty(G)$ is torsion-free. Moreover, if $f : G \rightarrow H$ is a group homomorphism from G to a torsion-free group H , then $\text{Tor}_\infty(G) \leq \ker(f)$.*

Proof. If $x^n \in \text{Tor}_\infty(G)$ for some $n > 0$, then there exists $m \in \mathbb{N}$ such that $x^n \in \text{Tor}_m(G)$. It follows that $x \in \text{Tor}_{m+1}(G)$, and thus that $x \in \text{Tor}_\infty(G)$. Thus $G/\text{Tor}_\infty(G)$ is torsion-free.

If $f : G \rightarrow H$ is a group homomorphism from G to a torsion-free group H , it follows that $\text{Tor}(G) \leq \ker(f)$, and thus that $\text{Tor}_1(G) \leq \ker(f)$. Then, f factors through $G/\text{Tor}_1(G)$. By induction, we see that $\text{Tor}_n(G) \leq \ker(f)$ for all n , and thus that $\text{Tor}_\infty(G) \leq \ker(f)$. \square

Remark 3.4. Let **Grp** be the category of groups, and let **TfGrp** be the category of torsion-free groups. Lemma 3.3 essentially says that the functor which sends a group G to $G/\text{Tor}_\infty(G)$ is left adjoint to the forgetful functor from **TfGrp** to **Grp**.

Since the conjugate of a torsion element is torsion, the normal subgroup generated by the set of torsion elements is the same as the subgroup generated by them. The following lemma records this, and other, important facts.

Lemma 3.5. *Let G be a group. Then, for all $i \in \mathbb{N}$, we have:*

- a.) $\text{Tor}_{i+1}(G) = \langle \{g \in G \mid g \text{Tor}_i(G) \in \text{Tor}(G/\text{Tor}_i(G))\} \rangle$
- b.) $\text{Tor}_{i+1}(G) = \langle \{g \in G \mid g^n \in \text{Tor}_i(G) \text{ for some } n > 0\} \rangle$
- c.) $\text{Tor}_{i+1}(G)/\text{Tor}_i(G) = \text{Tor}_1(G/\text{Tor}_i(G))$ as subgroups of $G/\text{Tor}_i(G)$
- d.) $(G/\text{Tor}_i(G))/\text{Tor}_1(G/\text{Tor}_i(G)) \cong G/\text{Tor}_{i+1}(G)$, via the obvious quotient map.

Proof. We leave these to the reader. \square

Lemma 3.6. *For any group G any any $i, j > 0$, we have*

$$(G/\text{Tor}_i(G))/\text{Tor}_j(G/\text{Tor}_i(G)) \cong G/\text{Tor}_{i+j}(G)$$

via the map $(g \text{Tor}_i(G)) \text{Tor}_j(G/\text{Tor}_i(G)) \mapsto g \text{Tor}_{i+j}(G)$.

Proof. This follows via induction over j , making use of lemma 3.5 d.) . \square

Lemma 3.7. *Let $H \leq G$ be groups. Then $\text{Tor}_i(H) \leq \text{Tor}_i(G)$ for all $i \in \mathbb{N}$.*

Proof. We induct over i . Clearly, $\text{Tor}_1(H) \leq \text{Tor}_1(G)$, as $\text{Tor}(H) \subseteq \text{Tor}(G)$. Now assume that $\text{Tor}_i(H) \leq \text{Tor}_i(G)$ for all $i \leq n$. Then

$$\begin{aligned} \text{Tor}_{n+1}(H) &= \langle \{g \in H \mid g^k \in \text{Tor}_n(H) \text{ for some } n > 0\} \rangle \text{ by lemma 3.5 b),} \\ &\leq \langle \{g \in H \mid g^k \in \text{Tor}_n(G) \text{ for some } n > 0\} \rangle \text{ by induction hypothesis,} \\ &\leq \text{Tor}_{n+1}(G) \text{ by lemma 3.5 b).} \end{aligned}$$

So the induction is complete. \square

The following is a standard result from combinatorial group theory.

Lemma 3.8. *Let $P = \langle X | R \rangle$ be a recursive presentation. Then the words in X^* which represent elements in the subgroup $\text{Tor}_i(\overline{P})$ are recursively enumerable, uniformly in P and in each $i \in \mathbb{N}$. Hence the words representing elements in $\text{Tor}_\infty(\overline{P})$ are also recursively enumerable.*

We make the following definition:

Definition 3.9. We define the *Torsion Length* of G , $\text{TorLen}(G)$, by the smallest n such that $\text{Tor}_n(G) = \text{Tor}_\infty(G)$ (equivalently, the smallest n such that $\text{Tor}_n(G) = \text{Tor}_{n+1}(G)$). If no such n exists, then we say G has infinite torsion length, and write $\text{TorLen}(G) = \infty$.

The same notion appears in Cirio *et. al.* [3] as the *Torsion Degree* ([3, Definition 5.5]) of a group.

Some basic properties of torsion length are:

Lemma 3.10.

- a.) *If G is a non-trivial torsion group (i.e. $\text{Tor}(G) = G$), then $\text{TorLen}(G) = 1$. In particular, all non-trivial finite groups have torsion length 1.*
- b.) *If $i \leq \text{TorLen}(G)$, then $\text{TorLen}(G/\text{Tor}_i(G)) = \text{TorLen}(G) - i$, with the convention $\infty - i = \infty$ and $\infty - \infty = 0$.*
- c.) *$\text{TorLen}(G)$ is the smallest i for which $G/\text{Tor}_i(G)$ is torsion-free (and ∞ if no such finite i exists).*

Proof. 1. is immediate, 2. follows from lemma 3.6, and 3. is immediate. \square

The following result shows that we retain some measure of control when considering the torsion of HNN extensions and free products with amalgamation.

Theorem 3.11 ([8, Theorem 11.69]). *Let $g \in \text{Tor}(G)$. Then:*

- 1. *If $G = K_1 *_H K_2$ is an amalgamated product, then g is conjugate to an element of K_1 or K_2 .*
- 2. *If $G = K *_H$ is an HNN extension, then g is conjugate to an element in the base group K .*

From this point on, if $\{A_i\}_{i \in I}$ is a family of groups, then we write $*_{i \in I} A_i$ to denote the free product of all the A_i . Using this, we state the following, the proof of which is very similar to that of theorem 3.11.

Corollary 3.12. *Suppose $g \in \text{Tor}(*_{i \in I} A_i)$, where I is any index set. Then g is conjugate to an element in one of the A_i 's.*

Proposition 3.13. *Let A, B be groups, and H a group which embeds into both A and B . Then $\text{Tor}_1(A *_H B) = \langle\langle \text{Tor}(A) \cup \text{Tor}(B) \rangle\rangle^{A *_H B}$ (where $\text{Tor}(A)$ and $\text{Tor}(B)$ are viewed as subsets of A, B respectively, and hence as subsets of $A *_H B$ under the natural embeddings).*

Proof. This is an immediate consequence of theorem 3.11. \square

Note. The above proposition holds only when considering $\text{Tor}_1(A *_H B)$, and not any higher $\text{Tor}_i(A *_H B)$ for $i > 1$. This is due to the fact that, once we

take the quotient by $\text{Tor}_1(A *_H B)$, there may be many more cancellations introduced into the group, and so one or both of the factors may collapse too soon.

We can extend the above proposition, if we choose to work with free products *without* amalgamation. Observe that the following result only holds for standard free products, and does *not* carry over to free products with amalgamation or HNN extensions.

Proposition 3.14. *Let $\{A_i\}_{i \in I}$ be a family of groups, and let $*_{i \in I} A_i$ denote their free product. Then, for all $j \in \mathbb{N}$, $\text{Tor}_j(*_{i \in I} A_i) = \langle\langle \cup_{i \in I} \text{Tor}_j(A_i) \rangle\rangle^{*_{i \in I} A_i}$, and the natural map*

$$*_{i \in I} (A_i / \text{Tor}_j(A_i)) \rightarrow (*_{i \in I} A_i) / \text{Tor}_j(*_{i \in I} A_i)$$

*is an isomorphism. Furthermore, $\text{Tor}_\infty(*_{i \in I} A_i) = \langle\langle \cup_{i \in I} \text{Tor}_\infty(A_i) \rangle\rangle^{*_{i \in I} A_i}$, and the natural map*

$$*_{i \in I} (A_i / \text{Tor}_\infty(A_i)) \rightarrow (*_{i \in I} A_i) / \text{Tor}_\infty(*_{i \in I} A_i)$$

is an isomorphism.

Proof. The fact that $\text{Tor}_1(*_{i \in I} A_i) = \langle\langle \cup_{i \in I} \text{Tor}_1(A_i) \rangle\rangle^{*_{i \in I} A_i}$ follows immediately from corollary 3.12. Using this, it follows easily that the natural map

$$*_{i \in I} (A_i / \text{Tor}_1(A_i)) \rightarrow (*_{i \in I} A_i) / \text{Tor}_1(*_{i \in I} A_i)$$

is an isomorphism. Using lemma 3.5, lemma 3.6, and induction, we see that $\text{Tor}_j(*_{i \in I} A_i) = \langle\langle \cup_{i \in I} \text{Tor}_j(A_i) \rangle\rangle^{*_{i \in I} A_i}$ for all j . The fact that the natural map

$$*_{i \in I} (A_i / \text{Tor}_j(A_i)) \rightarrow (*_{i \in I} A_i) / \text{Tor}_j(*_{i \in I} A_i)$$

is an isomorphism is an immediate consequence. The statement for Tor_∞ now follows by taking unions. \square

Corollary 3.15. *Let $\{A_i\}_{i \in I}$ be a family of groups. Then*

$$\text{TorLen}(*_{i \in I} A_i) = \sup\{\text{TorLen}(A_i)\}_{i \in I}$$

Remark 3.16. The last part of proposition 3.14 follows immediately from the fact that left adjoints preserve colimits. However, we have decided to give an explicit proof.

4. CONSTRUCTIONS OF GROUPS WITH ARBITRARY TORSION LENGTH

In this section, we provide examples of finitely presented groups of torsion length n , for arbitrary finite n . We begin by given an example of a family of groups with torsion length 2.

Proposition 4.1 ([2, Proposition 4.10]). *Given any $j, k, l > 1$, we can define the finite presentation*

$$P_{j,k,l} := \langle x, y, z \mid x^j = e, y^k = e, xy = z^l \rangle$$

Then $\overline{P}_{j,k,l} / \langle\langle \text{Tor}(\overline{P}_{j,k,l}) \rangle\rangle^{\overline{P}_{j,k,l}} \cong C_l$, and hence $\text{TorLen}(\overline{P}_{j,k,l}) = 2$.

Proof. As the value of the subscripts on $P_{j,k,l}$ is irrelevant for this argument, we suppress them. It is clear from the presentation P that $\overline{P} \cong (C_j * C_k) *_{\langle xy \rangle = \langle z^l \rangle} \mathbb{Z}$; the amalgamated product of $C_j * C_k$ and \mathbb{Z} over infinite cyclic subgroups. By proposition 3.13, $\text{Tor}_1(\overline{P}) = \langle\langle \text{Tor}(C_j * C_k) \rangle\rangle^{\overline{P}}$. Moreover, $x, y \in \text{Tor}(\overline{P})$, and so $\text{Tor}_1(\overline{P}) = \langle\langle C_j * C_k \rangle\rangle^{\overline{P}}$. So it follows that $\overline{P}/\text{Tor}_1(\overline{P})$ has finite presentation $Q := \langle x, y, z | x^j = e, y^k = e, xy = z^l, x = e, y = e \rangle$, hence $\overline{Q} \cong C_l$ which is not torsion-free but has torsion length 1. Thus $\text{TorLen}(\overline{P}) = 2$ by lemma 3.10. \square

We can now generalise the above example to construct finitely presented groups of torsion length n , for arbitrary $n \in \mathbb{N}$.

First, we need some notation for binary strings.

Definition 4.2. Let $\{0, 1\}^n$ denote the set of binary strings of length precisely n , where we define $\{0, 1\}^0 := \{\emptyset\}$. If $\eta \in \{0, 1\}^n$, then we write $\eta 0$ (respectively, $\eta 1$) for the binary string of length $n + 1$, given by appending 0 (respectively, 1) to the rightmost end of η . Moreover, if $\eta \in \{0, 1\}^n$, then we write η' for the binary string of length $n - 1$ given by removing the rightmost digit from η .

We thank Claudia Pinzari; her questions led us to the following generalisation of proposition 4.1.

Theorem 4.3. *There is a family of finite presentations $\{P_n\}_{n \in \mathbb{N}}$ of groups satisfying $\text{TorLen}(\overline{P}_n) = n$ and $\overline{P}_n/\text{Tor}_1(\overline{P}_n) \cong \overline{P}_{n-1}$. Explicitly, these are:*

$$P_n := \left\langle x_\eta \ \forall \eta \in \bigcup_{i=0}^{n-1} \{0, 1\}^i \mid x_\eta^3 = e \ \forall \eta \in \{0, 1\}^{n-1}, \ x_{\eta 0} x_{\eta 1} = x_\eta^3 \ \forall \eta \in \bigcup_{i=0}^{n-2} \{0, 1\}^i \right\rangle$$

which have $2^n - 1$ generators, and $2^n - 1$ relators.

Proof. We show this by explicit construction. Set $P_0 := \langle - | - \rangle$.

Now define $P_1 := \langle x_\emptyset | x_\emptyset^3 = e \rangle$. Then $\overline{P}_1 \cong C_3$, which has torsion length 1.

Now define $P_2 := \langle x_\emptyset, x_0, x_1 | x_\emptyset^3 = e, x_1^3 = e, x_0 x_1 = x_\emptyset^3 \rangle$. This is the example from proposition 4.1, and so \overline{P}_2 has torsion length 2. Note that $\overline{P}_2 \cong (C_3 * C_3) *_{x_0 x_1 = x_\emptyset^3} (\mathbb{Z}) \cong (\overline{P}_1 * \overline{P}_1) *_{x_0 x_1 = x_\emptyset^3} (\mathbb{Z})$.

Now, in general, for $n \in \mathbb{N}$, define the finite presentation

$$P_n := \left\langle x_\eta \ \forall \eta \in \bigcup_{i=0}^{n-1} \{0, 1\}^i \mid x_\eta^3 = e \ \forall \eta \in \{0, 1\}^{n-1}, \ x_{\eta 0} x_{\eta 1} = x_\eta^3 \ \forall \eta \in \bigcup_{i=0}^{n-2} \{0, 1\}^i \right\rangle$$

It can then be seen that $\overline{P}_{n+1} \cong (\overline{P}_n * \overline{P}_n) *_{\langle rs \rangle = \langle t^3 \rangle} (\mathbb{Z})$, where r, s are the generators of the final \mathbb{Z} factor in each copy of \overline{P}_n , and t is the generator of the \mathbb{Z} factor added to form \overline{P}_{n+1} . We now prove by induction that $\overline{P}_n/\text{Tor}_1(\overline{P}_n) \cong \overline{P}_{n-1}$: In proposition 4.1 we showed that $\overline{P}_2/\text{Tor}_1(\overline{P}_2) \cong \overline{P}_1$. Now suppose $\overline{P}_j/\text{Tor}_1(\overline{P}_j) \cong \overline{P}_{j-1}$ for all $j \leq n$. Writing $\overline{P}_{n+1} \cong (\overline{P}_n * \overline{P}_n) *_{\langle rs \rangle = \langle t^3 \rangle} (\mathbb{Z})$, we see (by two applications of proposition 3.13) that $\text{Tor}_1(\overline{P}_{n+1}) = \langle\langle \text{Tor}(\overline{P}_n * \overline{P}_n) \rangle\rangle^{\overline{P}_{n+1}} = \langle\langle \text{Tor}(\overline{P}_n) \cup \text{Tor}(\overline{P}_n) \rangle\rangle^{\overline{P}_{n+1}}$ (the notation here is unfortunate; $\text{Tor}(\overline{P}_n) \cup \text{Tor}(\overline{P}_n)$ denotes the union of the torsion elements of the two individual factors of $\overline{P}_n * \overline{P}_n$). Since $n \geq 1$, r and s remain non-trivial in their respective factors of $\overline{P}_n/\text{Tor}_1(\overline{P}_n)$ (though they may have finite order, and this will only occur

when $n = 1$), so $\langle rs \rangle$ is still infinite cyclic, so the amalgamation is unaffected. By the inductive hypothesis, $\overline{P}_n / \text{Tor}_1(\overline{P}_n) \cong \overline{P}_{n-1}$, so we have

$$\begin{aligned} \overline{P}_{n+1} / \text{Tor}_1(\overline{P}_{n+1}) &= (\overline{P}_n * \overline{P}_n) *_{\langle rs \rangle = \langle t^3 \rangle} (\mathbb{Z}) / \langle\langle \text{Tor}(\overline{P}_n) \cup \text{Tor}(\overline{P}_n) \rangle\rangle^{\overline{P}_{n+1}} \\ &\cong ((\overline{P}_n / \text{Tor}_1(\overline{P}_n)) * (\overline{P}_n / \text{Tor}_1(\overline{P}_n))) *_{\langle rs \rangle = \langle t^3 \rangle} (\mathbb{Z}) \\ &\cong (\overline{P}_{n-1} * \overline{P}_{n-1}) *_{\langle rs \rangle = \langle t^3 \rangle} (\mathbb{Z}) \\ &\cong \overline{P}_n \end{aligned}$$

which completes the inductive step.

Therefore, by lemma 3.10 (noting that $\overline{P}_{n+1} \not\cong \overline{P}_n$ for any n) we see that $\text{TorLen}(\overline{P}_{n+1}) = \text{TorLen}(\overline{P}_n) + 1$. By induction (recalling that $\text{TorLen}(\overline{P}_1) = 1$), it is immediate that $\text{TorLen}(\overline{P}_n) = n$. The number of generators and relators is self-evident. \square

The recursive definition $\overline{P}_{n+1} := (\overline{P}_n * \overline{P}_n) *_{\langle rs \rangle = \langle t^3 \rangle} (\mathbb{Z})$ first appeared (as far as we are aware) in [3, Example 5.16] by Cirio *et. al.* as a generalisation of [2, Proposition 4.10]. Our work is independent of that in [3], but given how natural the extension is, it is unsurprising that the two constructions are the same.

These examples that we have constructed, \overline{P}_n , are quite well behaved. We now show that they are word-hyperbolic, and thus enjoy all the properties of such groups, including (but not limited to): having solvable word, conjugacy, and isomorphism problem, having linear Dehn function, being biautomatic, being SQ-universal.

To do this, we need the following theorem by Kharlampovich-Myasnikov [5]. We thank Jack Button for pointing this out to us, and for suggesting that we might be able to show that the \overline{P}_n are word-hyperbolic via application of this result.

Theorem 4.4 ([5, Corollary 2]). *Let G_1, G_2 be word-hyperbolic groups, and $A \leq G_1$, $B \leq G_2$ virtually cyclic subgroups. Then the group $G_1 *_{A=B} G_2$ is word-hyperbolic if and only if either A is conjugate separated in G_1 or B is conjugate separated in G_2 .*

Proposition 4.5. *The groups P_n constructed in theorem 4.3 are word-hyperbolic, for all $n \in \mathbb{N}$. As a consequence, for every $n \in \mathbb{N}$, there exists a finitely presented word-hyperbolic group of torsion length n .*

Proof. We proceed by straightforward induction. Firstly, P_1 is word-hyperbolic, as it is finite. Now, $\overline{P}_{n+1} \cong (\overline{P}_n * \overline{P}_n) *_{\langle rs \rangle = \langle t^3 \rangle} (\mathbb{Z})$, where the notation is as in theorem 4.3. As \overline{P}_n has no elements of order 2 (by theorem 3.11), we see by lemma 2.4 that $\langle rs \rangle$ is conjugate separated in $\overline{P}_n * \overline{P}_n$. Since $\langle ab \rangle$ and $\langle t^3 \rangle$ are both cyclic, it follows by theorem 4.4 that \overline{P}_{n+1} is word-hyperbolic. This completes the induction. \square

Remark 4.6. In theorem 4.3, we chose to use exponent 3 in order to introduce torsion into \overline{P}_n . We could have chosen any exponent $k > 2$ and all proceeding results and proofs would still hold. However, in the case of exponent $k = 2$, we would lose the property of being word-hyperbolic; this can be shown by

combining the remark after lemma 2.4 with the ‘only if’ part of theorem 4.4. However, in the exponent-2 case, \bar{P}_1 and \bar{P}_2 are both solvable (proposition 7.2).

We now provide another perspective on these matters, using the following construction of Leary and Nucinkis [6] to create examples of groups of arbitrary torsion length.

Theorem 4.7 ([6, §5 Corollary 7]). *Let G be any group. There is a group \tilde{G} and a surjection $\phi : \tilde{G} \rightarrow G$ such that $\ker(\phi) = \text{Tor}_1(\tilde{G})$ (and so $\tilde{G}/\text{Tor}_1(\tilde{G}) \cong G$). A presentation for \tilde{G} can be formed from a presentation for G , with the use of 2 more generators and at most 2 more relators. Thus, if G is finitely generated (respectively, presented), then \tilde{G} can also be made to be finitely generated (respectively, presented) in a uniform algorithmic manner.*

In the above result, we note that \tilde{G} being finitely presented if G is does not follow immediately from the proof in [6, §5 Corollary 7]; a slight variant is required (pointed out to the authors by Ian Leary). For clarity, we state the modified proof for the finitely presented case here:

Proof. Let $\psi : F_k \rightarrow G$ be a surjection from a free group (of minimal rank) to G , with kernel N . So $N = \langle\langle R \rangle\rangle^{F_k}$ for some finite set $R \subset F_k$ (since G is finitely presented). Hence $\langle R \rangle^{F_k}$ is a free group of some finite rank $r := \text{rank}(\langle R \rangle^{F_k})$ (note that this need not be normal in F_k). The group $C_2 * C_3 := \langle x, y | x^2, y^3 \rangle$ contains an embedded copy of F_2 , freely generated by $a := yxy$ and $b := xyxyx$ (by lemma 2.1). Thus the subgroup of $C_2 * C_3$ generated by $S_n := \{b^{-1}ab, \dots, b^{-n}ab^n\}$ freely generates an embedded copy of F_n . Using Stallings foldings (see [4]) we can effectively compute $r := \text{rank}(\langle R \rangle^{F_k})$, as well as a free generating set $\{t_1, \dots, t_r\}$ of $\langle R \rangle^{F_k}$. Now form the free product with amalgamation

$$\tilde{G} := F_k *_\phi (C_2 * C_3)$$

where $\phi(t_i) = b^{-i}ab^i$ (so extends to an isomorphism between $\langle R \rangle^{F_k}$ and $\langle S_r \rangle^{C_2 * C_3}$). Note that $\text{Tor}_1(\tilde{G}) = \langle\langle x, y \rangle\rangle^{\tilde{G}}$, so annihilating the torsion of \tilde{G} leaves us exactly with G (annihilating $C_2 * C_3$ means we annihilate S_r , and hence R , and hence the normal closure of R , which is N). Now, recalling the substitution $a := yxy, b := xyxyx$, taking a free generating set $\{z_1, \dots, z_k\}$ for F_k , and writing each t_i in terms of the z_j ’s, we have that \tilde{G} is given by the finite presentation:

$$\langle z_1, \dots, z_k, x, y | x^2 = e, y^3 = e, t_i = b^{-i}ab^i \ \forall 1 \leq i \leq r \rangle$$

So we need a presentation with only two more generators, and at most 2 more relators, than a presentation of G (fewer if $|S_r| < |R|$, i.e. $\text{rank}(\langle R \rangle^{F_k}) < |R|$). \square

This allows us to show the following:

Corollary 4.8. *There is a sequence of finitely presented groups $\{G_n\}_{n \in \mathbb{N}}$ such that, for each n , $\text{TorLen}(G_n) = n$ and $G_n / \text{Tor}_1(G_n) \cong G_{n-1}$. Moreover, each G_n has a finite presentation (call this Q_n) with $2n - 1$ generators and at most $2n - 1$ relators.*

Proof. Set $G_1 := C_2$ with finite presentation $\langle z|z^2 \rangle$, and inductively define $G_{n+1} := \tilde{G}_n$ for each $n > 1$. The result then follows from theorem 4.7 and its proof. \square

It is unclear whether these groups are word-hyperbolic; theorem 4.4 cannot be used here, as the amalgamated subgroup is almost certainly not virtually cyclic.

Remark 4.9. Theorem 4.7 complements theorem 4.3 in a nice fashion. Both results give examples of finitely presented groups of arbitrary finite torsion length; however, the respective constructions differ in certain aspects. While the finite presentations Q_n for G_n (as in corollary 4.8) above have $2n-1$ generators and at most $2n-1$ relators, the relators themselves are extremely complicated (the authors struggled to write down an explicit finite presentation for G_3). On the other hand, though the groups constructed in theorem 4.3 (the P_n) have 2^n-1 generators and 2^n-1 relators, the generators and relators are conveniently indexed, and all relators have length 3 or 5. It is useful to note that there are indeed finite presentations of groups of arbitrary torsion length (the Q_n 's) such that the number of generators/relators is linear in the torsion length.

From this we ask the following natural question:

Question. *Is there some finite bound k such that, for each $n \in \mathbb{N}$, there is a finite presentation with at most k generators and k relators of a group with torsion length n ?*

5. EMBEDDINGS

The following result is well-known (see, for example, [8, Corollary 11.72]). We choose to describe the construction in detail, as we will need it later.

Lemma 5.1. *There is a uniform procedure that, on input of any countably generated recursive presentation $P = \langle X|R \rangle$, outputs a 2-generator recursive presentation (denoted $\text{fg}(P)$) such that \overline{P} embeds in $\overline{\text{fg}(P)}$.*

Proof. Fix an enumeration x_1, x_2, \dots of all letters in X . Let $P_1 := \langle a, b | - \rangle$ be a presentation for the free group F_2 . Consider the following two subgroups of $\overline{P} * \overline{P}_1$:

$$A := \langle a, x_1 b^{-1} a b, x_2 b^{-2} a b^2, \dots, x_i b^{-i} a b^i, \dots \rangle$$

and

$$B := \langle b, a^{-1} b a, \dots, a^{-i} b a^i, \dots \rangle$$

Note that the sets $\{a^{-i} b a^i\}_{i \in \mathbb{N}}$ and $\{b^{-i} a b^i\}_{i \in \mathbb{N}}$ freely generate copies of F_∞ in \overline{P}_1 . Thus, by the normal form theorem for free products, $\{x_i b^{-i} a b^i\}_{i \in \mathbb{N}}$ freely generates a copy of F_∞ in $\overline{P} * \overline{P}_1$, regardless of which w_i are/aren't trivial in \overline{P} . Thus A and B are isomorphic, and such an isomorphism can be given by the extension $\overline{\phi}$ of the set map $\phi : \{x_i b^{-i} a b^i\}_{i \in \mathbb{N}} \rightarrow \{a^{-i} b a^i\}_{i \in \mathbb{N}}$; $\phi(x_i b^{-i} a b^i) := a^{-i} b a^i$ for all $i \in \mathbb{N}$. We now form the HNN extension $\overline{P} *_\phi$ of \overline{P} , conjugating A to B : This can be realised via the following presentation:

$$Q := \langle X, a, b, t | R, t^{-1} x_i b^{-i} a b^i t = a^{-i} b a^i \forall i \geq 0 \rangle$$

It is not hard to see that \overline{Q} is generated by a and t . Removing X and b from the generating set of Q , and making the relevant substitutions in the relating set of Q gives us our desired 2-generator recursive presentation, which we denote by $\text{fg}(P)$; by construction it is then clear that \overline{P} embeds in $\overline{\text{fg}(P)}$. \square

Lemma 5.2. *In the construction given in lemma 5.1, if instead the input P is a finite presentation, then the output $\text{fg}(P)$ can be made to be a 2-generator finite presentation such that \overline{P} embeds in $\overline{\text{fg}(P)}$.*

Proof. Follow the proof of lemma 5.1, we simply observe that if $P = \langle X|R \rangle$ has generating set $X = \{x_1, \dots, x_n\}$, then we form the HNN extension by identifying the two subgroups

$$A := \langle a, x_1 b^{-1} a b, x_2 b^{-2} a b^2, \dots, x_n b^{-n} a b^n \rangle$$

and

$$B := \langle b, a^{-1} b a, \dots, a^{-n} b a^n \rangle$$

where $A \cong B \cong F_{n+1}$. This HNN extension can be realised via the following presentation (where $x_0 := \emptyset$):

$$Q := \langle X, a, b, t | R, t^{-1} x_i b^{-i} a b^i t = a^{-i} b a^i \ \forall 0 \leq i \leq n \rangle$$

Since R is a finite set, Q is thus a (2-generator) finite presentation (all details are as in the proof of lemma 5.1). \square

In a slight abuse of notation, we will write $\text{fg}(P)$ to denote:

1. The construction given in lemma 5.1, if P is a countably generated recursive presentation.
2. The construction given in lemma 5.2, if P is a finite presentation.

Lemma 5.3. *Let $P = \langle X|R \rangle$ be a recursive (alternatively, finite) presentation. Take any recursive enumeration (alternatively, finite collection) S of words in X^* . Then $\text{fg}(\langle X|R \cup S \rangle)$ is the presentation $\text{fg}(\langle X|R \rangle)$ with S adjoined to its relating set.*

Proof. The construction of $\text{fg}(\langle X|R \rangle)$ is completely uniform in the relating set R . Thus we can add relators either before or after the amalgamation step, and it does not change the final presentation. \square

Of course, in the above result we need to be careful about the notion of the union of two recursive enumerations of elements, as a recursive enumeration.

Corollary 5.4. *Let $P = \langle X|R \rangle$ be a recursive (alternatively, finite) presentation. Take any recursive enumeration (alternatively, finite collection) S of words in X^* . Then*

$$\overline{\text{fg}(\langle X|R \cup S \rangle)} \cong \overline{\text{fg}(P)} / \langle\langle S \rangle\rangle^{\overline{\text{fg}(P)}}$$

Corollary 5.5. *Let $P = \langle X|R \rangle$ be a recursive presentation. Take an enumeration T_i of all words in X^* representing elements in $\text{Tor}_i(\overline{P})$ (lemma 3.8). Then*

$$\overline{\text{fg}(\langle X|R \cup T_i \rangle)} \cong \overline{\text{fg}(P)} / \text{Tor}_i(\overline{\text{fg}(P)})$$

That is to say, taking the i^{th} torsion quotient of P and then applying the fg process gives the same group as first applying the fg process and then taking the i^{th} torsion quotient; the two commute.

Lemma 5.6. *Let P be a countably generated recursive presentation. Then $\overline{\text{fg}(P)}$ is torsion-free if and only if \overline{P} is torsion-free*

Proof. By an application of theorem 3.11 to the construction of $\text{fg}(P)$ (as an HNN extension), every torsion element in $\overline{\text{fg}(P)}$ is conjugate to a torsion element in \overline{P} (when \overline{P} is viewed with the natural embedding in to $\overline{\text{fg}(P)}$). \square

We now give the main result of this section.

Theorem 5.7. *Let P be a countably generated recursive presentation (respectively, finite presentation). Then we can construct a 2-generator recursive presentation (respectively, finite presentation) $\text{fg}(P)$ as given in lemma 5.1 (respectively, lemma 5.2), uniformly in P , such that \overline{P} embeds in $\overline{\text{fg}(P)}$, and $\text{TorLen}(\overline{\text{fg}(P)}) = \text{TorLen}(\overline{P})$.*

Proof. The first part of the theorem is proved in lemma 5.1 (respectively, lemma 5.2). All that remains to be shown is that $\overline{\text{TorLen}(\text{fg}(P))} = \overline{\text{TorLen}(P)}$. By corollary 5.5, for any $i \in \mathbb{N}$, we have that $\overline{\text{fg}(\langle X|R \cup T_i \rangle)} \cong \overline{\text{fg}(P) / \text{Tor}_i(\text{fg}(P))}$ (where T_i is an enumeration of all words in X^* representing elements in $\text{Tor}_i(\overline{P})$, via lemma 3.8). By lemma 5.6, $\overline{\text{fg}(\langle X|R \cup T_i \rangle)}$ is torsion-free if and only if $\overline{\langle X|R \cup T_i \rangle}$ is. Since $\overline{P} / \text{Tor}_i(\overline{P}) \cong \overline{\langle X|R \cup T_i \rangle}$, we get that $\overline{\text{fg}(P) / \text{Tor}_i(\text{fg}(P))}$ is torsion-free if and only if $\overline{P} / \text{Tor}_i(\overline{P})$ is. By lemma 3.10, $\text{TorLen}(\overline{P})$ is the smallest i such that $\overline{P} / \text{Tor}_i(\overline{P})$ is torsion-free, which in turn is the smallest i such that $\overline{\text{fg}(P) / \text{Tor}_i(\text{fg}(P))}$ is torsion-free (by what we have just shown), which in turn is $\text{TorLen}(\overline{\text{fg}(P)})$ (by lemma 3.10 again). \square

In lemma 3.7 we showed that $\text{Tor}_i(H) \leq \text{Tor}_i(G)$ whenever $H \leq G$. Unfortunately, this cannot be extended to the preservation of torsion length under subgroups or quotients, even for finitely presented groups. To show this fact, we first need the following two results in combinatorial group theory:

Theorem 5.8 (Adian-Rabin). *There is a uniform construction that, for each finite presentation $P = \langle X|R \rangle$ of a group and each $w \in X^*$, outputs a finite presentation $P(w)$ of a group and an explicit homomorphism $\phi : \overline{P} \rightarrow \overline{P(w)}$ such that:*

1. *If $w \neq e$ in \overline{P} , then $\phi : \overline{P} \hookrightarrow \overline{P(w)}$ is an embedding.*
2. *If $w = e$ in \overline{P} , then $\overline{P(w)} \cong \{e\}$.*

Moreover, $\overline{P(w)}$ is the normal closure of w , and so $\overline{P(w)} / \langle\langle w \rangle\rangle^{\overline{P(w)}} \cong \{e\}$.

Proof. A description of this is given in [1, Theorem 2.4]. \square

Theorem 5.9 (Higman). *There is a uniform algorithm that, on input of a countably generated recursive presentation $P = \langle X|R \rangle$, constructs a finite presentation $T(P)$ such that $\overline{P} \hookrightarrow \overline{T(P)}$, along with an explicit map ϕ which extends to an embedding $\overline{\phi} : \overline{P} \hookrightarrow \overline{T(P)}$.*

Proof. A description of this is given in [1, Lemma 6.9] and [1, Theorem 6.10]. \square

Proposition 5.10. *Given a recursively presented group G , and any $k > 0$, there exists some 2-generator finitely presented group H with torsion length 1 into which G embeds.*

Proof. We first show this for finitely presented groups. Let P be a finite presentation of a group. Form the free product presentation $Q := P * \langle z | z^2 \rangle$, so $\overline{Q} \cong \overline{P} * C_2$. Now use theorem 5.8 to construct the finite presentation $Q(z)$. Since $z \neq e$ in \overline{Q} , we have that $\overline{P} \hookrightarrow \overline{Q} \hookrightarrow \overline{Q(z)}$. But $w \in \text{Tor}(\overline{Q}) \subseteq \overline{Q(w)}$, so $\overline{Q(w)}/\text{Tor}_1(\overline{Q(w)}) \cong \{e\}$ by the last line in theorem 5.8. Hence $\text{TorLen}(\overline{Q(w)}) = 1$ (as $\overline{Q(w)}$ is non-trivial), and $\overline{Q(w)}$ contains an embedded copy of \overline{P} .

Now if A is a countably generated recursive presentation, form the finite presentation $T(A)$ as in theorem 5.9. Then use the above argument to embed $\overline{T(A)}$ into a finitely presented group with torsion length 1. \square

Corollary 5.11. *Given $m, n > 0$ there are finitely presented groups G, H with torsion lengths m, n respectively such that $H \leq G$. Thus torsion length is in no way preserved under subgroups.*

Proof. Take the finite presentations P_i from theorem 4.3, with $\text{TorLen}(\overline{P}_i) = i$. Take P_n , and use proposition 5.10 to embed this into a finitely presented group (with finite presentation B) of torsion length 1. This in turn embeds into the group given by the finite presentation $B * P_m$, which has torsion length m (by corollary 3.15 and theorem 4.3). \square

Lemma 5.12. *For each $k > 0$, there is a 2-generator finitely presented group H_k with torsion length k .*

Proof. Take the finite presentation P_k of a group with torsion length k from theorem 4.3, and use theorem 5.7 to embed it into a 2-generator group given by the finite presentation $\text{fg}(P_k)$, with torsion length k . \square

Corollary 5.13. *Let H_k be a 2-generator finitely presented group, as in the preceding lemma. For each $m, n > 0$, F_4 surjects onto $F_2 * H_n$ (of torsion length n), which in turn surjects onto F_2 , which in turn surjects onto H_m (of torsion length m). Thus torsion length is in no way preserved under quotients.*

Proof. As H_k is 2-generator for each k , it is a quotient of F_2 . The corollary then follows from corollary 3.15. \square

6. A 2-GENERATOR GROUP WITH INFINITE TORSION LENGTH

Recall that a *recursive presentation* is a group presentation $\langle X | R \rangle$, where X is a finite set, and R is a recursive enumeration of relators (elements of X^*).

Lemma 6.1. *Take the finite presentations P_0, P_1, \dots from theorem 4.3. Form their free product presentation $P := P_0 * P_1 * \dots$. Then*

$$\overline{P}/\text{Tor}_1(\overline{P}) \cong \overline{P}$$

Proof. Using proposition 3.14 and theorem 4.3 we get that

$$\begin{aligned} \overline{P}/\text{Tor}_1(\overline{P}) &= (\overline{P}_1 * \overline{P}_2 * \dots)/\text{Tor}_1(\overline{P}_1 * \overline{P}_2 * \dots) \\ &\cong (\overline{P}_1/\text{Tor}_1(\overline{P}_1)) * (\overline{P}_2/\text{Tor}_1(\overline{P}_2)) * \dots \\ &\cong \{e\} * \overline{P}_1 * \overline{P}_2 * \dots \\ &\cong \overline{P} \end{aligned}$$

□

Note that alternatively we could have taken the finite presentations Q_n from theorem 4.8 in the above proof; the proof would follow in an identical manner.

Observing that \overline{P} as constructed above is not torsion-free (so $\text{Tor}_1(\overline{P})$ is non-trivial), \overline{P} is thus non-Hopfian; the surjective non-injective map given by taking the quotient by $\text{Tor}_1(\overline{P})$. Thus we get the following immediate corollary:

Corollary 6.2. *With P as above, $\text{TorLen}(\overline{P}) = \infty$.*

Theorem 6.3. *There exists a 2-generator recursive presentation Q for which $\text{TorLen}(\overline{Q}) = \infty$.*

Proof. Take the countably generated recursive presentation P from lemma 6.1, for which $\text{TorLen}(\overline{P}) = \infty$ by corollary 6.2. But theorem 5.7 gives that $\text{TorLen}(\overline{\text{fg}(P)}) = \text{TorLen}(\overline{P}) (= \infty)$. So taking $Q := \text{fg}(P)$ gives a 2-generator recursive presentation of a group \overline{Q} with $\text{TorLen}(\overline{Q}) = \infty$. □

Such a group must necessarily have a non-trivial torsion-free quotient:

Proposition 6.4. *Let G be a finitely generated group. If $\text{TorLen}(G) = \infty$, then $\text{Tor}_\infty(G) \neq G$ (as a subset of G). Equivalently, $G/\text{Tor}_\infty(G)$ is non-trivial, and thus G has a non-trivial torsion-free (universal) quotient.*

Proof. By definition, $\text{Tor}_\infty(G) := \bigcup_{i \in \mathbb{N}} \text{Tor}_i(G)$, where $\text{Tor}_1(G) \leq \text{Tor}_2(G) \leq \dots$. If $\text{Tor}_\infty(G) = G$, then any finite generating set for G will lie in $\text{Tor}_n(G)$ for some finite n . Hence $\text{Tor}_n(G) = G$, and so $\text{Tor}_n(G) = \text{Tor}_\infty(G)$. Thus $\text{TorLen}(G) \leq n < \infty$, a contradiction. □

Compare this with the observation that \overline{P} , as constructed in lemma 6.1, is such that $\text{Tor}_\infty(\overline{P}) = \overline{P}$. That is, \overline{P} has a trivial universal torsion-free quotient $\overline{P}/\text{Tor}_\infty(\overline{P})$, even though $\overline{P}/\text{Tor}_n(\overline{P}) \cong \overline{P}$ for every finite n .

Of course, the most natural question to ask at this point is:

Question. *Does there exist a finitely presented group of infinite torsion length?*

Perhaps more ambitiously, is there a finitely presented (or even finitely generated) group G with torsion for which $G/\text{Tor}_1(G) \cong G$? Such a group would be non-Hopfian, with a non-injective surjection given by taking the quotient by $\text{Tor}_1(G)$.

7. TORSION LENGTH FOR SOLVABLE GROUPS

Proposition 7.1. *Let G be a group for which $\text{Tor}(G)$ forms a subgroup. Then $\text{TorLen}(G) \leq 1$.*

Proof. Since $\text{Tor}(G)$ is a subgroup, it is immediate that $\text{Tor}(G) = \text{Tor}_1(G)$ (by lemma 3.5). Suppose $x \text{Tor}_1 G$ is a torsion element of $G/\text{Tor}_1(G)$. Then, there exists an n such that $x^n \in \text{Tor}_1(G)$. However, since $\text{Tor}_1(G) = \text{Tor}(G)$, it follows that there exists an m such that $x^{mn} = e$. Thus x is torsion in G , and we have that $G/\text{Tor}_1(G)$ is torsion-free. Thus $\text{TorLen}(G) \leq 1$. \square

Groups in which the set of torsion elements forms a subgroup include abelian groups (obviously) and nilpotent groups ([7, 5.2.7]). In this section, we show that it is possible for a solvable group to have torsion length greater than 1. Recall the family of groups constructed in proposition 4.1:

$$P_{j,k,l} := \langle x, y, z \mid x^j = e, y^k = e, xy = z^l \rangle.$$

Proposition 7.2. *The group $\overline{P}_{j,k,l}$ is solvable if and only if $j = k = l = 2$. Moreover, $\overline{P}_{2,2,2}$ is polycyclic.*

Proof. Since $C_j * C_k$ embeds into $\overline{P}_{j,k,l}$, it follows from lemma 2.1 that $\overline{P}_{j,k,l}$ is not solvable if either j or k is not 2. Suppose $j = k = 2$. Then, it is not hard to see that $\overline{P}_{2,2,l}$ surjects onto $C_2 * C_l$ (to see this, introduce the relation $z^l = e$ in to the presentation $P_{2,2,l}$). Again, lemma 2.1 tells us that l must be 2 if $\overline{P}_{2,2,l}$ were to be solvable.

We now show $\overline{P}_{2,2,2}$ is polycyclic. It follows from the presentation $P_{2,2,2}$ that the subgroup generated by z^2 is normal. Quotienting out by this subgroup, we get $C_2 * C_2$, which is polycyclic by lemma 2.2. Thus $\overline{P}_{2,2,2}$ is polycyclic. \square

Corollary 7.3. *There exists a polycyclic group of torsion length 2.*

While we suspect there exist solvable groups of arbitrary finite torsion length, we have been unable to construct them. A related interesting question is:

Question. *Does there exist a finitely generated solvable groups of infinite torsion length?*

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